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An analytical proof of a certain geometric inequality conjecture of Shan-He Wu

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ABSTRACT

In this short note, the authors give a proof of a certain geometric inequality conjecture of Shan-He Wu by making use of some analytical techniques (see Srivastava et al. (2011) [5] and Wu et al. (2010) [6]). Finally, three closely-related geometric inequality problems are posed as open problems.

1. Introduction and main results

For a given triangle $\triangle ABC$, let a, b and c denote the side-lengths facing the angles A, B and C, respectively. Also let s denote the semi-perimeter and r the inradius of the triangle $\triangle ABC$. Moreover, we will customarily use the cyclic sum and product symbols, namely,

$$\sum f(a) = f(a) + f(b) + f(c),$$
$$\sum f(b, c) = f(a, b) + f(b, c) + f(c, a),$$
$$\prod f(a) = f(a)f(b)f(c),$$

and so on.

It is easy to prove the following inequality by means of the familiar *Power Mean Inequality*:

$$\sqrt{2}(a + b + c) \leqq \sum \sqrt{b^2 + c^2}. \tag{1.1}$$

By making use of *different* methods, Zhou [1] and Wu [2] proved the reverse of the inequality (1.1) as follows:

$$\sum \sqrt{b^2 + c^2} < \left(1 + \frac{\sqrt{2}}{2}\right)(a + b + c) \tag{1.2}$$

where the constant $1 + \frac{\sqrt{2}}{2}$ in the inequality (1.2) is the best possible. On the other hand, Wu [2] considered refinement of the inequality (1.2) and presented the following geometric inequality conjecture involving the side-lengths and inradius of a given triangle $\triangle ABC$.

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Conjecture. *For a given triangle $\triangle ABC$, prove or disprove the following inequality:*

$$\sum \sqrt{b^2+c^2} \leqq \left(1+\frac{\sqrt{2}}{2}\right)(a+b+c) - 3\sqrt{3}(2-\sqrt{2})r. \tag{1.3}$$

Analytic as well as geometric inequalities are potentially useful in many different areas of the mathematical, physical and engineering sciences (see, for details, [3,4]; see also the recent investigations by Srivastava et al. [5] and Wu et al. [6]). With this objective in view, we present here an analytical proof of the inequality (1.3).

2. Preliminary lemmas

In order to prove the inequality (1.3), we require the following lemmas associated with a given triangle $\triangle ABC$.

Lemma 1. *If $a \leqq b \leqq c$, then*

$$\sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c) \leqq \frac{\sqrt{6}}{4} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.1}$$

Proof. From the following equivalence:

$$b^2+c^2 \geqq \frac{1}{2}(b+c)^2 \Longleftrightarrow \sqrt{b^2+c^2} \geqq \frac{\sqrt{2}}{2}(b+c),$$

we easily find that

$$\begin{aligned} \sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c) &= \frac{b^2+c^2-\frac{1}{2}(b+c)^2}{\sqrt{b^2+c^2}+\frac{\sqrt{2}}{2}(b+c)} \\ &= \frac{(b-c)^2}{2\sqrt{b^2+c^2}+\sqrt{2}(b+c)} \\ &\leqq \frac{(b-c)^2}{2\sqrt{2}(b+c)}. \end{aligned} \tag{2.2}$$

We now prove the following inequality:

$$\frac{(b-c)^2}{2\sqrt{2}(b+c)} \leqq \frac{\sqrt{6}}{4} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}, \tag{2.3}$$

which obviously is equivalent to the inequality given below:

$$\frac{1}{b+c} \leqq \sqrt{3} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{1}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.4}$$

Since

$$a = (s-b)+(s-c) \geqq 2\sqrt{(s-b)(s-c)}, \tag{2.5}$$

we have

$$\frac{\sqrt{3}}{2a} \cdot \sqrt{\frac{s-a}{s}} \leqq \sqrt{3} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{1}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.6}$$

By the hypothesis $a \leqq b \leqq c$, we obtain $b+c-2a \geqq 0$. We thus obtain

$$\begin{aligned} \frac{3(s-a)}{4a^2 s} - \frac{1}{(b+c)^2} &= \frac{3(b+c-a)}{4a^2(a+b+c)} - \frac{1}{(b+c)^2} \\ &= \frac{(b+c-2a)[2a^2+3(b+c)a+3(b+c)^2]}{4a^2(b+c)^2(a+b+c)} \geqq 0. \end{aligned}$$

Therefore, we have

$$\frac{1}{(b+c)^2} \leqq \frac{3(s-a)}{4a^2 s} \Longleftrightarrow \frac{1}{b+c} \leqq \frac{\sqrt{3}}{2a} \cdot \sqrt{\frac{s-a}{s}}. \tag{2.7}$$

From the inequalities (2.6) and (2.7), we know that the inequality (2.4) holds true, that is, the inequality (2.3) holds true. Applying the inequalities (2.2) and (2.3), we immediately get the inequality (2.1). □

Lemma 2. *If $a \leqq b \leqq c$, then*

$$\sqrt{c^2+a^2}+\sqrt{a^2+b^2}-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} \leqq \frac{\sqrt{6}}{8}\cdot\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.8}$$

Proof. Let

$$u=\sqrt{b^2+c^2}, \qquad v=\sqrt{c^2+a^2} \qquad \text{and} \qquad w=\sqrt{a^2+b^2}.$$

Then, by using the hypothesis $a \leqq b \leqq c$, we have

$$v=\sqrt{c^2+a^2} \leqq \sqrt{2}c \qquad \text{and} \qquad w=\sqrt{a^2+b^2} \leqq \sqrt{2}b.$$

We thus find that

$$\begin{aligned}(v+w)^2 &= 2(v^2+w^2)-(v-w)^2\\ &= 2(2a^2+b^2+c^2)-\frac{(v^2-w^2)^2}{(v+w)^2}\\ &= 2(2a^2+b^2+c^2)-\frac{(b+c)^2(b-c)^2}{(v+w)^2}\\ &\leqq 2(2a^2+b^2+c^2)-\frac{(b+c)^2(b-c)^2}{2(b+c)^2}\\ &= 2(2a^2+b^2+c^2)-\frac{1}{2}(b-c)^2\\ &= 4a^2+(b+c)^2+\frac{1}{2}(b-c)^2.\end{aligned}$$

Hence

$$\sqrt{c^2+a^2}+\sqrt{a^2+b^2} \leqq \sqrt{4a^2+(b+c)^2+\frac{1}{2}(b-c)^2}. \tag{2.9}$$

By applying the inequality (2.9), we now have

$$\begin{aligned}\sqrt{c^2+a^2}+\sqrt{a^2+b^2}-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} &\leqq \sqrt{4a^2+(b+c)^2+\frac{1}{2}(b-c)^2}-\sqrt{4a^2+(b+c)^2}\\ &= \frac{\frac{1}{2}(b-c)^2}{\sqrt{4a^2+(b+c)^2+\frac{1}{2}(b-c)^2}+\sqrt{4a^2+(b+c)^2}}\\ &\leqq \frac{(b-c)^2}{4\sqrt{4a^2+(b+c)^2}},\end{aligned} \tag{2.10}$$

which, in view of the inequality (2.5), yields

$$\frac{\sqrt{6}}{16}\cdot\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a} \leqq \frac{\sqrt{6}}{8}\cdot\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.11}$$

We next prove the following inequality:

$$\frac{(b-c)^2}{4\sqrt{4a^2+(b+c)^2}} \leqq \frac{\sqrt{6}}{16}\cdot\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a}$$

or, equivalently,

$$\frac{1}{\sqrt{4a^2+(b+c)^2}} \leqq \frac{\sqrt{6}}{4}\cdot\sqrt{\frac{s-a}{s}}\cdot\frac{1}{a}. \tag{2.12}$$

For $a \leqq b \leqq c$, we have $b+c-2a \geqq 0$, and so we get

$$\begin{aligned}\frac{3(s-a)}{8a^2 s} - \frac{1}{4a^2+(b+c)^2} &= \frac{3(b+c-a)}{8a^2(a+b+c)} - \frac{1}{4a^2+(b+c)^2} \\ &= \frac{(b+c-2a)[10a^2+3(b+c)a+3(b+c)^2]}{8a^2(a+b+c)[4a^2+(b+c)^2]} \geqq 0,\end{aligned}$$

which implies that

$$\frac{1}{4a^2+(b+c)^2} \leqq \frac{3(s-a)}{8a^2 s}. \tag{2.13}$$

The inequality (2.12) now follows immediately from the inequality (2.13).

Finally, by appealing to the inequalities (2.10) to (2.12), we can conclude that the inequality (2.8) holds true. □

Lemma 3. *If* $a \leqq b \leqq c$ *and*

$$f(a,b,c) := \sum \sqrt{b^2+c^2} - \left(1+\frac{\sqrt{2}}{2}\right)(a+b+c) + 3\sqrt{3}(2-\sqrt{2}) \cdot \frac{\sqrt{s \cdot \prod(s-a)}}{s},$$

then

$$f(a,b,c) \leqq f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right). \tag{2.14}$$

Proof. First of all, the inequality (2.14) can be stated in the following equivalent form:

$$\begin{aligned}&\sum \sqrt{b^2+c^2} - \left(1+\frac{\sqrt{2}}{2}\right)(a+b+c) + 3\sqrt{3}(2-\sqrt{2}) \cdot \frac{\sqrt{s \cdot \prod(s-a)}}{s} \\ &\leqq \frac{\sqrt{2}}{2}(b+c) + 2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} - \left(1+\frac{\sqrt{2}}{2}\right)(a+b+c) + \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot a\sqrt{\frac{s-a}{s}}\end{aligned}$$

or

$$\begin{aligned}&\sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c) + \sqrt{c^2+a^2} + \sqrt{a^2+b^2} - 2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} \\ &\leqq \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot \sqrt{\frac{s-a}{s}} \cdot \left[a - 2\sqrt{(s-b)(s-c)}\right].\end{aligned} \tag{2.15}$$

Furthermore, the inequality (2.15) is equivalent to the following inequality:

$$\begin{aligned}&\sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c) + \sqrt{c^2+a^2} + \sqrt{a^2+b^2} - 2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} \\ &\leqq \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}.\end{aligned} \tag{2.16}$$

Thus, by applying Lemmas 1 and 2, we get

$$\begin{aligned}&\sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c) + \sqrt{c^2+a^2} + \sqrt{a^2+b^2} - 2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} \\ &\leqq \frac{3\sqrt{6}}{8} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}.\end{aligned} \tag{2.17}$$

It is now fairly obvious that

$$\frac{3\sqrt{6}}{8} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}} \leqq \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.18}$$

The inequality (2.16) follows from the inequalities (2.17) and (2.18). Consequently, the inequality (2.15) holds true. We can thus conclude that the inequality (2.14) holds true. The proof of Lemma 3 is evidently completed. □

Lemma 4 (*See [7,8]*). *Given the following polynomial $f(x)$ with real coefficients:*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

if the number of the sign changes of the revised sign list of its discriminant sequence:

$$\{D_1(f), D_2(f), \ldots, D_n(f)\}$$

is v, then the number of the pairs of distinct conjugate imaginary roots of $f(x)$ equals v. Furthermore, if the number of non-vanishing members of the revised sign list is l, then the number of distinct real roots of $f(x)$ equals $l - 2v$.

3. Proof of the conjectured inequality (1.3)

In this section, we present our analytical proof of the conjectured inequality (1.3).

Since the inequality (1.3) is symmetrical with respect to the side-lengths a, b and c, there is no harm in supposing that $a \leqq b \leqq c$. Thus, by Lemma 3, we only need to prove that

$$f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) \leqq 0$$

or, equivalently, that

$$\frac{\sqrt{2}}{2}(b+c) + 2\sqrt{a^2 + \left(\frac{b+c}{2}\right)^2} - \left(1 + \frac{\sqrt{2}}{2}\right)(a+b+c) + \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot a\sqrt{\frac{s-a}{s}} \leqq 0. \tag{3.1}$$

Without loss of generality, we can assume that

$$a = 1 \quad \text{and} \quad \frac{b+c}{2} = x \quad (x \geqq 1),$$

because the inequality (3.1) is homogeneous with respect to a and $\frac{b+c}{2}$. Thus, clearly, the inequality (3.1) is equivalent to the following inequality:

$$2\sqrt{1+x^2} - 2x - \left(1 + \frac{\sqrt{2}}{2}\right) + \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot \sqrt{\frac{2x-1}{2x+1}} \leqq 0 \quad (x \geqq 1). \tag{3.2}$$

Define the function $g(x)$ by

$$g(x) := 2\sqrt{1+x^2} - 2x - \left(1 + \frac{\sqrt{2}}{2}\right) + \frac{3\sqrt{3}(2-\sqrt{2})}{2} \cdot \sqrt{\frac{2x-1}{2x+1}} \quad (x \geqq 1).$$

Calculating the first two derivatives of $g(x)$ with respect to x, we get

$$g'(x) = \frac{2x}{\sqrt{1+x^2}} - 2 + 3\sqrt{3}(2-\sqrt{2}) \cdot \frac{1}{(2x+1)^2} \cdot \sqrt{\frac{2x+1}{2x-1}}$$

and

$$g''(x) = \frac{2}{(1+x^2)\sqrt{1+x^2}} - \frac{6\sqrt{3}(2-\sqrt{2})(4x-1)}{(2x-1)(2x+1)^2\sqrt{(2x-1)(2x+1)}},$$

respectively. By setting $g''(x) = 0$, we obtain

$$(2x-1)(2x+1)^2\sqrt{(2x-1)(2x+1)} - 3\sqrt{3}(2-\sqrt{2})(4x-1)(1+x^2)\sqrt{1+x^2} = 0. \tag{3.3}$$

It is easy to see that the roots of Eq. (3.3) are also solutions of the following equation:

$$(2x-1)^3(2x+1)^5 - 27(6-4\sqrt{2})(4x-1)^2(1+x^2)^3 = 0,$$

that is,

$$\frac{-73+54\sqrt{2}}{503} \cdot p(x) = 0, \tag{3.4}$$

where

$$\begin{aligned} p(x) = {} & 16096x^8 + (20736\sqrt{2} + 19984)x^7 - (49248\sqrt{2} + 17282)x^6 + (10368\sqrt{2} - 10128)x^5 \\ & - (8262 + 44064\sqrt{2})x^4 + (23328\sqrt{2} + 7392)x^3 - (2494 + 15984\sqrt{2})x^2 \\ & + (1004 + 6696\sqrt{2})x - (235 + 918\sqrt{2}). \end{aligned}$$

The revised sign list of the discriminant sequence of $p(x)$ is given by

$$[1, 1, 1, -1, -1, 1, 1, -1]. \tag{3.5}$$

So the number of the sign changes of the revised sign list of (3.5) is 3. Thus, by applying Lemma 4, we find that the following equation:

$$p(x) = 0 \tag{3.6}$$

has two distinct real roots. Moreover, it is not difficult to observe that

$$p(-5) = 4481713395 - 2452789998\sqrt{2} > 0,$$

$$p(-4) = 664401285 - 565132086\sqrt{2} < 0,$$

$$p(1) = 6075 - 49086\sqrt{2} < 0$$

and

$$p(2) = 5167125 - 735750\sqrt{2} > 0.$$

We can thus find that Eq. (3.6) has two distinct real roots in the following intervals:

$$(-5, -4) \quad \text{and} \quad (1, 2),$$

so that Eq. (3.6) has only one real root x_0 given by

$$x_0 = 1.474289581\cdots$$

on the interval $[1, +\infty)$. It is also easy to find that

$$g''(x) > 0 \qquad \text{when} \qquad x \in (x_0, +\infty)$$

and that

$$g''(x) < 0 \qquad \text{when} \qquad x \in (1, x_0).$$

Hence $g'(x)$ is strictly monotone decreasing on the interval $(1, x_0)$ and strictly monotone increasing on the interval $(x_0, +\infty)$ (see Fig. 1). From the limit relation:

$$\lim_{x \to 1^+} g'(x) = 0,$$

together with the limit relation:

$$\lim_{x \to +\infty} g'(x) = 0,$$

we can conclude that

$$g'(x) < 0 \qquad \text{when} \qquad x \in (1, +\infty).$$

Consequently, $g(x)$ is strictly monotone decreasing on the interval $[1, +\infty)$. Thus, obviously, we have

$$g(x) \leqq g(1) = 0,$$

that is, the inequality (3.2) or (3.1) holds true. Our analytical proof of the conjectured inequality (1.3) is completed.

Remark. Fig. 1 was drawn by means of the computer software *Maple* (Version 9.0).

4. A set of open problems

In this concluding section, we propose three *open* problems. We first recall that, in the year 2003, Ye [9] generalized the inequality (1.2) as follows:

$$\sum \sqrt[n]{b^n + c^n} < \left(1 + \frac{\sqrt[n]{2}}{2}\right)(a + b + c) \quad (n \in \mathbb{N} \setminus \{1\}), \tag{4.1}$$

where $\mathbb{N}$ denotes the set of *positive* integers.

Considering the refinement of the inequality (4.1), we propose the following open problem.

Problem 1. For a given triangle $\triangle ABC$, if $n \in \mathbb{N} \setminus \{1, 2\}$, then prove or disprove the following inequality:

$$\sum \sqrt[n]{b^n + c^n} \leqq \left(1 + \frac{\sqrt[n]{2}}{2}\right)(a + b + c) - 3\sqrt{3}(2 - \sqrt[n]{2})r. \tag{4.2}$$

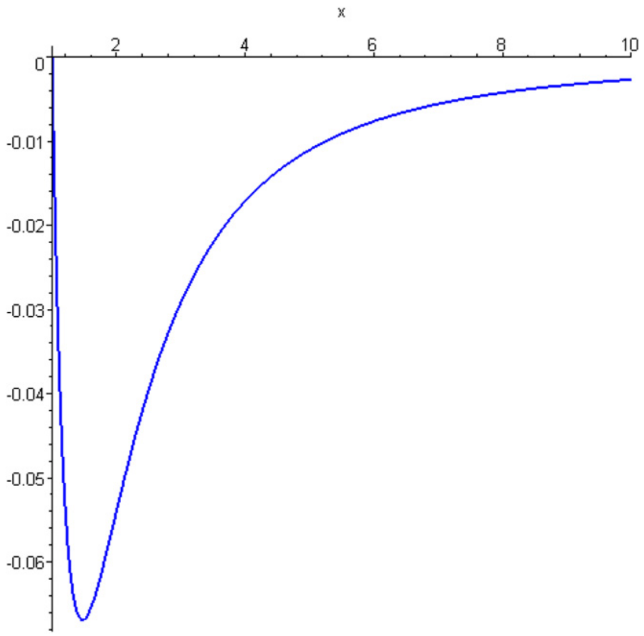

Fig. 1. The image of $g'(x)$.

Next, considering the analogous inequality of (1.2) in a tetrahedron, we propose another open problem as follows.

Problem 2. Let a_i $(i = 1, \ldots, 6)$ denote the lengths of the edges of a given tetrahedron *ABCD*. Also let ρ be the inradius of the tetrahedron. Then determine the best constant k such that the following inequality holds true:

$$\sum_{1\leqq i<j\leqq 6} \sqrt{a_i^2 + a_j^2} \leqq k \sum_{i=1}^{6} a_i. \tag{4.3}$$

If the best constant k for the inequality (4.3) is denoted by k_0, then our third open problem can be stated as follows.

Problem 3. For a given tetrahedron *ABCD*, prove or disprove the following inequality:

$$\sum_{1\leqq i<j\leqq 6} \sqrt{a_i^2 + a_j^2} \leqq k_0 \sum_{i=1}^{6} a_i - 6\sqrt{6}(2k_0 - 5\sqrt{2})\rho. \tag{4.4}$$

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